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Time-Optimal Rotational Motion

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SUMMARY
Pontryagin's maximum principle is used to solve the problem of bringing a rotating rigid body to rest in minimum time.

1. Introduction

Application of the Pontryagin maximum principle leads, in most problems of practical interest, to a two-point boundary-value problem with its attendant computational difficulties. It is important in such cases to have rapid methods of obtaining near-optimal solutions, and also to obtain an estimate of the error in such solutions.

The problem of bringing a rotating rigid body to rest in minimum time is used as an example to show how the method of "backing out of the origin" may be used to obtain a number of optimal trajectories. (We shall refer to trajectories obtained in this way as *trial trajectories.*) An initial set of trial trajectories is used as a guide to obtaining further trial trajectories, the latter passing closer to the desired initial point in state space. This process is repeated, if necessary, until a set of initial values of the adjoint variables, already used in a trial trajectory, can be combined with the given initial set of state variables to give an acceptably accurate near-optimal trajectory.

2. Formulation of the Problem

The general rotational motion of a rigid body is described by the equations

$$A\dot{p} - (B-C)qr = L\,, \tag{1}$$
$$B\dot{q} - (C-A)rp = M\,, \tag{2}$$
$$C\dot{r} - (A-B)pq = N\,, \tag{3}$$

where A, B, C are the principal moments of inertia at the centre of mass, p, q, r are the components of angular velocity about the corresponding principal axes, and L, M, N are the externally applied moments about these axes. The initial conditions are

$$t = t_0, \quad p = p_0, \quad q = q_0, \quad r = r_0\,,$$

and the final conditions are

$$t = t_1, \quad p = q = r = 0\,.$$

It is assumed that L, M, N are bounded, as follows:

$$|L| \leqq L_m\,, \tag{4}$$
$$M_l \leqq M \leqq M_u\,, \tag{5}$$
$$|N| \leqq N_m\,, \tag{6}$$

where $M_l < 0$ and $M_u > 0$. This type of control is common on aircraft, where the ailerons and rudder are bounded symmetrically and the elevator is bounded asymmetrically. (It is hoped later to increase the number and complexity of the equations of motion so that they will represent, for example, a spinning aircraft.)

The problem to be considered is that of finding L, M, N such that the transfer time $t_1 - t_0$ is minimised.

3. Application of the Maximum Principle

It is convenient to write equations (1), (2), (3) in the form

$$\dot{p} = aqr + u_1 , \tag{7}$$
$$\dot{q} = brp + u_2 , \tag{8}$$
$$\dot{r} = cpq + u_3 , \tag{9}$$

where

$$a = (B-C)/A, \quad b = (C-A)/B, \quad c = (A-B)/C ,$$
$$u_1 = L/A, \quad u_2 = M/B, \quad u_3 = N/C .$$

In this notation, equations (4), (5), (6) are written

$$|u_1| \leqq u_{1m} , \tag{10}$$
$$u_{2l} \leqq u_2 \leqq u_{2u} , \tag{11}$$
$$|u_3| \leqq u_{3m} . \tag{12}$$

In order to apply the Pontryagin maximum principle, [1], [2], we require the variational Hamiltonian

$$\mathscr{H}(\boldsymbol{\lambda}, \boldsymbol{\omega}, \mathbf{u}) \equiv \lambda_0 + \lambda_1(aqr + u_1) + \lambda_2(brp + u_2) + \lambda_3(cpq + u_3) , \tag{13}$$

where

$$\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \lambda_2, \lambda_3), \quad \boldsymbol{\omega} = (p, q, r), \quad \mathbf{u} = (u_1, u_2, u_3) .$$

The maximum principle then gives the following necessary conditions for optimality.

If $\mathbf{u}^*(t)$, for $t_0 \leqq t \leqq t_1$, is an optimal control, then there exists a nonzero, continuous function $\boldsymbol{\lambda}(t)$, which is a solution of the adjoint equations

$$\dot{\lambda}_1 = -br\lambda_2 - cq\lambda_3 , \tag{14}$$
$$\dot{\lambda}_2 = -cp\lambda_3 - ar\lambda_1 , \tag{15}$$
$$\dot{\lambda}_3 = -aq\lambda_1 - bp\lambda_2 , \tag{16}$$

and which satisfies the conditions

$$\sup_{u \in U} \mathscr{H}(\boldsymbol{\lambda}(t), \boldsymbol{\omega}^*(t), \mathbf{u}) = \mathscr{H}(\boldsymbol{\lambda}(t), \boldsymbol{\omega}^*(t), \mathbf{u}^*(t)) = 0 , \tag{17}$$

$$\lambda_0(t) = \text{constant} \leqq 0 , \tag{18}$$

U being the set of admissible controls determined by equations (10), (11), (12).

From equations (17) and (13),

$$\lambda_0 + \lambda_1(aq^* r^* + u_1^*) + \lambda_2(br^* p^* + u_2^*) + \lambda_3(cp^* q^* + u_3^*) = 0 , \tag{19}$$

$$u_1^* = u_{1m} \operatorname{sgn} \lambda_1 , \tag{20}$$
$$u_2^* = \tfrac{1}{2}[(u_{2l} + u_{2u}) - (u_{2l} - u_{2u}) \operatorname{sgn} \lambda_2] , \tag{21}$$
$$u_3^* = u_{3m} \operatorname{sgn} \lambda_3 . \tag{22}$$

Since the state variables p, q, r are fixed at $t = t_0$ and at $t = t_1$, the initial and final values of the adjoint variables are free, i.e. neither the initial nor final conditions for equations (14), (15), (16) are known. Furthermore, since these equations are homogeneous in λ_1, λ_2, λ_3, we can assume without loss of generality that the final values of the variables satisfy

$$|\lambda_1(t_1)| + |\lambda_2(t_1)| + |\lambda_3(t_1)| = 1 . \tag{23}$$

In principle, all the quantities appearing in the calculation may now be determined. In particular, we may from now on ignore equation (19), since it may be regarded as merely determining the value of λ_0.

The equations to be solved are therefore (7), (8), (9), (14), (15), (16) and (20), (21), (22). The last three of these equations are used to eliminate u_1, u_2, u_3 from the first three. However, it is still not possible to solve directly the resulting set of six equations, because we have two-point boundary-values for p, q, r and no boundary-values for λ_1, λ_2, λ_3.

4. Optimal Trajectories

In the situation described in the previous section it is always possible to obtain any number of optimal trajectories by reversing the time variable [3] in equations (7), (8), (9) and (14), (15), (16). Specifically, we write

$$\tau = t_1 - t \tag{24}$$

the new time variable τ being the "time to go." The equations become

$$\dot{p} = -aqr - u_1 , \tag{25}$$
$$\dot{q} = -brp - u_2 , \tag{26}$$
$$\dot{r} = -cpq - u_3 \tag{27}$$

and

$$\dot{\lambda}_1 = br\lambda_2 + cq\lambda_3 , \tag{28}$$
$$\dot{\lambda}_2 = cp\lambda_3 + ar\lambda_1 , \tag{29}$$
$$\dot{\lambda}_3 = aq\lambda_1 + bp\lambda_2 . \tag{30}$$

(The dots now mean $d/d\tau$.) The "initial" conditions are now

$$\tau = 0, \quad p = q = r = 0 ,$$
$$|\lambda_1(0)| + |\lambda_2(0)| + |\lambda_3(0)| = 1 , \tag{31}$$

and the "final" conditions become

$$\tau = t_1 - t_0, \quad p = p_0, \quad q = q_0, \quad r = r_0 . \tag{32}$$

Suppose $\lambda_1(0)$, $\lambda_2(0)$, $\lambda_3(0)$ satisfying equation (31), are given. Then equations (25)–(30) may be solved by numerical integration, starting at $\tau = 0$, since u_1, u_2, u_3 are given by equations (20), (21), (22).

Unfortunately, it is most unlikely that the required set of "final" conditions (32) will be attained after $\lambda_1(0)$, $\lambda_2(0)$, $\lambda_3(0)$ have been chosen more or less arbitrarily. The fundamental difficulty is that the state equations (7), (8), (9) are nonlinear. For the case of linear state equations, Neustadt [4], [5], [6] has shown that the two-point boundary-value problem can be transformed into the problem of locating the point where a function of several variables takes its maximum value. Several methods are available for solving the latter problem, [7], [8].

The method used in the present paper for solving equations (25)–(30), while at the same time obtaining almost correct "final" conditions, is first to vary $\lambda_1(0)$, $\lambda_2(0)$, $\lambda_3(0)$ systematically, thus generating a set of trial trajectories. The "final" values p_0, q_0, r_0 on these trajectories are then used as a guide for further adjustments of $\lambda_1(0)$, $\lambda_2(0)$, $\lambda_3(0)$.

It was hoped, finally, to interpolate for the correct values of $\lambda_1(0)$, $\lambda_2(0)$, $\lambda_3(0)$ using the end-points of these trial trajectories and the given initial values p_0, q_0, r_0. However, no completely satisfactory interpolation formulas could be found. Instead, the final trajectory is a near-optimal trajectory, obtained by integrating equations (7), (8), (9) and (14), (15), (16) forwards, the initial λ-values being determined from the nearest trial trajectory.

The accuracy of the method can always be increased, of course, by computing more trial trajectories. A measure of the final accuracy is given by the minimum value of the function

$$f = p^2 + q^2 + r^2 \tag{33}$$

which, ideally, should be zero.

If f_a, f_b, f_c are the values of f evaluated at three equally spaced times t_a, t_b, t_c, then quadratic interpolation gives

$$f_{\min}=f_b-\frac{(f_a-f_c)^2}{8(f_a-2f_b+f_c)}, \tag{34}$$

a formula which is sufficiently accurate for the present purpose, provided that

$$|t_{\min}-t_b|/\Delta t \ll 1 ,$$

where Δt is the constant time interval. The time $t_{\min}$ is given by

$$t_{\min}=t_b+\frac{(f_a-f_c)\Delta t}{2(f_a-2f_b+f_c)}. \tag{35}$$

5. Numerical Example

In equations (1), (2), (3) take

$$A:B:C=3:8:10 .$$

Then, in equations (7), (8), (9),

$$a=-0.667, \quad b=0.875, \quad c=-0.500 .$$

The bounds on the controls are taken to be

$$u_{1m}=0.40, \quad u_{2l}=-0.20, \quad u_{2u}=0.13, \quad u_{3m}=0.14 ,$$

and are intended to be representative values for an orthodox aircraft.

After using equations (20), (21), (22), equations (25), (26), (27) become

$$\dot{p}=0.667qr-0.40 \operatorname{sgn} \lambda_1 , \tag{36}$$
$$\dot{q}=-0.875rp+0.035-0.165 \operatorname{sgn} \lambda_2 , \tag{37}$$
$$\dot{r}=0.500pq-0.14 \operatorname{sgn} \lambda_3 . \tag{38}$$

Changing the sign of each term on the right-hand side of these equations gives the numerical form of equations (7), (8), (9).

Rather than choose specific values for p_0, q_0, r_0, a more general investigation of the method was carried out. A set of trial trajectories were first computed, using the values ± 0.8 ± 0.1, ± 0.1 for $\lambda_1(0), \lambda_2(0), \lambda_3(0)$ in all possible combinations. Note that these values satisfy equation

TABLE 1

Initial trial trajectories

Case	$\lambda_1(0)$	$\lambda_2(0)$	$\lambda_3(0)$	$\tau=5$		
				p	q	r
1	0.8	0.1	0.1	−2.061	0.106	0.746
2	0.8	0.1	−0.1	−2.071	0.499	−0.716
3	0.8	−0.1	0.1	−2.019	0.469	0.162
4	0.8	−0.1	−0.1	−2.162	−0.050	−0.713
5	0.1	0.8	0.1	−1.908	0.985	−0.085
6	0.1	0.8	−0.1	−1.917	0.954	−0.397
7	0.1	0.1	0.8	−2.048	0.283	0.741
8	0.1	0.1	−0.8	−2.113	0.067	−0.880
9	0.1	−0.1	0.8	0.717	−0.504	−0.732
10	0.1	−0.1	−0.8	−2.120	−0.462	−0.845
11	0.1	−0.8	0.1	−1.902	−1.034	0.376
12	0.1	−0.8	−0.1	−1.827	−1.119	0.048

(31). This choice of "initial" λ's gives 24 trial trajectories altogether, though examination of equations (25)–(30) shows that only half of these need actually be computed; changing the signs of both $\lambda_1(0)$ and $\lambda_3(0)$ is equivalent to changing the signs of both p and r.

The computation was performed on an Elliott 4130 computer, using FORTRAN IV and a Runge-Kutta subroutine to solve the differential equations. The time variable τ was taken from 0–5 secs. in steps of 0.1, and the total computing time for the 12 cases was about 5 minutes. The values of p, q, r at $\tau=5$ are shown in Table 1.

Similar results for the remaining 12 cases are easily written down. With the exception of Case 9, p decreases steadily from zero to the neighbourhood of -2. In these circumstances, the interpolation of further trial trajectories is relatively easy.

The switching sequences are, of course, known for these trial trajectories, since the adjoint variables λ_1, λ_2, λ_3 are computed at the same time as the state variables, p, q, r.

Six more trial trajectories were interpolated between Cases 1 and 5. The values of p, q, r at $\tau=5$ for these trajectories are shown in Table 2.

TABLE 2

Further trial trajectories

Case	$\lambda_1(0)$	$\lambda_2(0)$	$\lambda_3(0)$	$\tau=5$		
				p	q	r
25	0.7	0.2	0.1	−1.977	0.705	0.568
26	0.6	0.3	0.1	−1.915	0.934	0.276
27	0.5	0.4	0.1	−1.904	0.977	0.127
28	0.4	0.5	0.1	−1.904	0.987	0.024
29	0.3	0.6	0.1	−1.905	0.987	−0.034
30	0.2	0.7	0.1	−1.907	0.985	−0.076

By interpolating between two cases in this way, and repeating the process, if necessary, a small region of (p, q, r)-space can be saturated with trial trajectories. Initial values of λ_1, λ_2, λ_3 are then available for the integration of equations (14), (15), (16) and (7), (8), (9).

To illustrate the effects of small errors in the initial values of λ_1, λ_2, λ_3 the following 52 near-optimal trajectories were computed, and the errors estimated by means of equation (34). Taking Case 1 above as the basic case, increments of ± 0.01 (or zero) were applied to p_0, q_0, r_0 in all possible ways (26 cases), and increments of ± 0.02 (or zero) were similarly applied. Equations (7), (8), (9) and (14), (15), (16) were solved for these 52 cases, using

$$\lambda_1(0)=0.874, \quad \lambda_2(0)=-0.029, \quad \lambda_3(0)=-0.185,$$

these being the "final" values of the λ's from the trial trajectory of Case 1. The results are shown in Table 3.

The symbols $+$, $-$, 0 denote the signs of the increments δp_0, δq_0, δr_0 to

$$(p_0, q_0, r_0) = (-2.061, 0.106, 0.746),$$

and

$$\omega_{\min} = (f_{\min})^{\frac{1}{2}},$$

where $f_{\min}$ is given by equation (34), with

$$t_a = 4.90, \quad t_b = 5.00, \quad t_c = 5.10.$$

Also, from equation (35), $\omega_{\min}$ occurs at time

$$t_{\min} = 5 + \frac{f_a - f_c}{20(f_a - 2f_b + f_c)},$$

which in all cases differs only very slightly from 5. The suffixes 1 and 2 on $\omega_{\min}$ indicate that the increments in p_0, q_0, r_0 are ± 0.01 and ± 0.02 respectively.

In only 6 of these 26 pairs of cases can it be said that the error in the final value of ω increases linearly with the initial error (as measured by δp_0, δq_0, δr_0), viz. Cases g, h, j, l, q, r. Sometimes an increase in the initial error leads to a decrease in the final error (Cases a, t).

TABLE 3

Final errors in 52 near-optimal trajectories

Case	δp_0	δq_0	δr_0	$(\omega_{min})_1$	$(\omega_{min})_2$
a	+	0	0	.021	.014
b	0	+	0	.009	.034
c	0	0	+	.009	.028
d	0	+	+	.015	.044
e	+	0	+	.008	.026
f	+	+	0	.031	.079
g	+	+	+	.036	.074
h	−	0	0	.023	.045
i	0	−	0	.039	.063
j	0	0	−	.009	.018
k	0	−	−	.043	.057
l	−	0	−	.026	.051
m	−	−	0	.061	.091
n	−	−	−	.064	.097
o	0	+	−	.011	.032
p	0	−	+	.037	.059
q	−	0	+	.022	.044
r	+	0	−	.020	.040
s	+	−	0	.018	.021
t	−	+	0	.013	.011
u	+	+	−	.029	.073
v	+	−	+	.018	.020
w	−	+	+	.014	.020
x	+	−	−	.023	.033
y	−	+	−	.017	.021
z	−	−	+	.059	.089

An attempt was made to find formulas for $\delta\lambda_1(0)$, $\delta\lambda_2(0)$, $\delta\lambda_3(0)$ in terms of p_0, q_0, r_0 and δp_0, δq_0, δr_0. However, no *consistently* reliable formulas have been found, and it is concluded that the values of $\lambda_1(0)$, $\lambda_2(0)$, $\lambda_3(0)$ which should be used as initial conditions for equations (14), (15), (16) are those from the nearest trial trajectory, i.e. the one for which

$$(\delta p_0)^2+(\delta q_0)^2+(\delta r_0)^2$$

is least.

6. Conclusions

The classical two-point boundary-value problem which occurs in the solution of optimisation problems may sometimes be avoided by the device of reversing the time variable in the governing differential equations. It is shown that this method may be used successfully in a practical problem (time-optimal rotational motion) when it is relatively easy to compute reversed optimal trajectories, and an exact solution is not required.

The error in the final near-optimal trajectory is assessed by determining how closely the given terminal conditions are satisfied. Since the method depends on constructing trial trajectories which steadily approach the required optimal trajectory, the errors in the final trajectory can, with sufficient computation, be made arbitrarily small.

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